

LAGRANGE ANCHOR FOR BARGMANN-WIGNER EQUATIONS

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ABSTRACT. A Poincaré invariant Lagrange anchor is found for the non-Lagrangian relativistic wave equations of Bargmann and Wigner describing free massless fields of spin $s > 1/2$ in four-dimensional Minkowski space. By making use of this Lagrange anchor, we assign a symmetry to each conservation law.

INTRODUCTION

The notions of symmetry and conservation law are of paramount importance for classical and quantum field theory. For Lagrangian theories both the notions are tightly connected to each other due to Noether's first theorem. Beyond the scope of Lagrangian dynamics, this connection has remained unclear, though many particular results and generalizations are known (see [1] for a review). In our recent works [2, 3] a general method has been proposed for connecting symmetries and conservation laws in not necessarily Lagrangian field theories. The key ingredient of the method is the notion of a Lagrange anchor introduced earlier [4] in the context of quantization of (non-)Lagrangian dynamics. Geometrically, the Lagrange anchor defines a map from the vector bundle dual to the bundle of equations of motion to the tangent bundle of the configuration space of fields such that certain compatibility conditions are satisfied. The existence of the Lagrange anchor is much less restrictive for the equations than the requirement to be variational or admit an equivalent variational reformulation.

The theory of massless higher-spin fields is an area of particular interest for application of the Lagrange anchor construction. Here one can keep in mind Vasiliev's higher-spin equations in the form of unfolded representation [5, 6, 7]. The unfolded field equations are not Lagrangian even at the free level and their quantization by the conventional methods is impossible. Finding a Lagrange anchor for these equations can be considered as an important step towards the consistent quantum theory of higher-spin fields. In our recent paper [8], a general construction for the Lagrange anchor was proposed for unfolded equations that admit an equivalent Lagrangian formulation.

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In this paper, the general concept of Lagrange anchor is exemplified by the Bargmann-Wigner equations for free massless fields of spin $s \geq 1/2$ in the four-dimensional Minkowski space [9]. The choice of the example is not accidental. First of all, it has long been known that the model admits infinite sets of symmetries and conservation laws. These have been a subject of intensive studies by many authors during decades, see e.g. [10, 11, 12, 13, 14, 15, 16, 17] and references therein. However, a complete classification has been obtained only recently, first for the conservation laws [18] and then for the symmetries [19]. As the field equations are non-Lagrangian for $s > 1/2$, there is no immediate Noether's correspondence between symmetries and conservation laws. The rich structure of symmetries and conservation laws in the absence of a Lagrangian formulation makes this theory an appropriate area for testing the concept of Lagrange anchor.

1. THE LAGRANGE ANCHOR IN FIELD THEORY

This section gives a brief exposition of the Lagrange anchor construction. A more detailed discussion can be found in [4].

The field equations are usually given by non-linear differential operators acting on the fields

$$T_a(\phi^i) = 0. \quad (1)$$

Here we use the condensed notation, so that indices i and a include local coordinates x on some space-time manifold X . In this notation, the partial derivatives with respect to fields should be understood as variational ones and summation by the repeating index includes integration by X . It is convenient to think that the left hand sides of equations possess their values in an infinite-dimensional vector bundle $\mathcal{E} \rightarrow M$ over the space of fields M . For the Lagrangian equations $T_i = \partial_i S$, where S is an action, the fields and equations are labeled by the same index $a = i$, and $\mathcal{E} = T^*M$. For non-Lagrangian equations $\mathcal{E} \neq T^*M$, and the discrete part of the index a may be different from that of i . We allow the fermionic degrees of freedom. The notation ϵ_a is used for the Grassmann parity of equation T_a .

Given the equations (1), a Lagrange anchor $V = (V_a^i(\phi))$ is defined to be a solution to the equation

$$V_a^i \partial_i T_b - (-)^{\epsilon_a \epsilon_b} V_b^i \partial_i T_a = C_{ab}^d T_d \quad (2)$$

where $C_{ab}^d(\phi)$ are some structure functions and the partial derivatives act on the left. The usual variational equations $T_i = \partial_i S$ admit an identical (or canonical) Lagrange anchor $V_i^j = \delta_i^j$. If the Lagrange anchor is invertible, then V^{-1} coincides with an integrating multiplier in the inverse problem of variational calculus. In this case, one can define a local action functional $S[\phi]$ such that $\partial_i S = (V^{-1})_i^a T_a$. For general field equations the Lagrange anchor may be given by a

differential operator $V : \mathcal{E}^* \rightarrow TM$ which is not necessarily invertible. A lot of examples of non-canonical Lagrange anchors for non-Lagrangian theories can be found in [2, 4, 8, 20, 21, 22, 23].

The vector field $j^\mu(x, \phi, \partial\phi, \dots)$ is called a conserved current if its divergence is proportional to the equations of motion, i.e.,

$$\int_X \partial_\mu j^\mu = \Psi^a T_a. \quad (3)$$

Here summation by a also includes the integration by the space-time manifold X . The coefficients Ψ^a are given by some differential operators with the values in \mathcal{E}^* and called characteristics. Two conserved currents j and j' are considered as equivalent if $j^\mu - (j')^\mu = \partial_\nu i^{\nu\mu} \pmod{T_a}$ for some bi-vector $i^{\mu\nu} = -i^{\nu\mu}$. Similarly, two characteristics Ψ and Ψ' are said to be equivalent if they correspond to equivalent currents. These equivalences can be used to simplify the form of characteristics. Namely, integrating by parts, one can remove all space-time derivatives of the field equations from the r.h.s of (3). Then the characteristic Ψ^a is simply given by a local function of fields and their derivatives. It can be shown that there is a one-to-one correspondence between equivalence classes of conserved currents and characteristics [2].

Given a Lagrange anchor, one can assign to any characteristic Ψ a variational vector field $X = (\Psi^a V_a^i) \partial_i$ on M . This vector field is a symmetry of system (1) because the transformation $\delta_X \phi^i = X^i$ preserves the equations of motion

$$\delta_X T_a \equiv X^i \partial_i T_a = 0 \quad (\text{mod } T_a).$$

The correspondence $\Psi^a \mapsto \Psi^a V_a^i$ between characteristics and symmetries, provided by the Lagrange anchor, was first introduced in [4]. Later it has been recognized that this correspondence has a natural interpretation in terms of BRST-theory [3]. In the variational case, the canonical Lagrange anchor connects characteristics with the symmetries of the action functional,

$$X^i T_i = 0 \quad \Leftrightarrow \quad \delta_X S = 0,$$

which is the statement of Noether's first theorem. For the general equations of motion and Lagrange anchors, the map from the space of characteristics to the space of symmetries is neither surjective nor injective. The symmetries from the image of a Lagrange anchor are called *characteristic symmetries*.

The space of symmetries has a lot of trivial elements vanishing on shell. They exist in any classical theory, being irrelevant to the structure of dynamics. The Lagrange anchor takes a trivial characteristic to a trivial symmetry, so one can systematically ignore the trivial elements in both spaces.

In order for the characteristic symmetries to form a subalgebra in the Lie algebra of all symmetries the following conditions must be satisfied:

$$[V_a, V_b] = C_{ab}^d V_d, \quad (-1)^{\epsilon_a \epsilon_c} (C_{ab}^d C_{cd}^e + V_c^i \partial_i C_{ab}^e) + \text{cycle}(a, b, c) = 0. \quad (4)$$

Here the commutator of variational vector fields is understood in the graded sense. The Lagrange anchor obeying (4) is called *strongly integrable*. The pull-back of the Lie bracket on the space of characteristic symmetries via the anchor map makes the linear space of characteristics into the Lie algebra. This Lie algebra structure is a generalization to the not necessarily Lagrangian dynamics of the Dickey bracket on the space of conserved currents [2].

2. THE LAGRANGE ANCHOR AND CHARACTERISTIC SYMMETRIES FOR THE BARGMANN-WIGNER EQUATIONS

In this section we illustrate the general concept of Lagrange anchor by the example of Bargmann-Wigner's equations. These equations describe free massless fields of spin $s > 0$ on $d = 4$ Minkowski space. The equations read

$$T_{\alpha_1 \dots \alpha_{2s-1}}^{\dot{\alpha}} := \partial^{\alpha \dot{\alpha}} \varphi_{\alpha \alpha_1 \dots \alpha_{2s-1}} = 0, \quad (5)$$

where $\varphi_{\alpha_1 \dots \alpha_{2s}}(x)$ is a symmetric, complex-valued spin-tensor field on $\mathbb{R}^{3,1}$. We use the standard notation of the two-component spinor formalism [9], e.g. $\partial^{\alpha \dot{\alpha}} = (\sigma^\mu)^{\alpha \dot{\alpha}} \partial / \partial x^\mu$, $\mu = 0, 1, 2, 3$, $\alpha, \dot{\alpha} = 1, 2$, and the spinor indices are raised/lowered with $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot{\alpha}\dot{\beta}}$ and the inverse $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot{\alpha}\dot{\beta}}$. The fields of integer spin are considered bosonic, and the half-integer spin fields are fermionic.

To make contact with the general definitions of the previous section let us mention that the indices of equations and fields take on the values $a = (\dot{\alpha} \alpha_1 \dots \alpha_{2s-1}, x^\mu)$, $i = (\alpha_1 \dots \alpha_{2s}, x^\mu)$.

In [23], it is shown that the Bargmann-Wigner equations for spin s admit the following Pioncaré-invariant and strongly integrable Lagrange anchor:

$$\begin{aligned} V_{\gamma_1 \dots \gamma_{2s-1}, \dot{\beta} \dot{\beta}_1 \dots \dot{\beta}_{2s-1}}^{\dot{\gamma}}(z, y) &= \\ &= i^{2s} \delta_{(\dot{\beta}}^{\dot{\gamma}} (\partial^z)_{\gamma_1 \dot{\beta}_1} \dots (\partial^z)_{\gamma_{2s-1} \dot{\beta}_{2s-1}}) \delta(z - y). \end{aligned} \quad (6)$$

The round brackets mean symmetrization. This Lagrange anchor is unique (up to equivalence) if the requirements of (i) field-independence, (ii) Pioncaré-invariance and (iii) locality are imposed. Being independent of fields, the Lagrange anchor is integrable and all the structure functions C_{ab}^d vanish.

Let Q be a characteristic of a conserved current j such that

$$\partial^{\alpha \dot{\alpha}} j_{\alpha \dot{\alpha}} = \Psi_{\dot{\alpha}}^{\alpha_1 \dots \alpha_{2s-1}} T_{\alpha_1 \dots \alpha_{2s-1}}^{\dot{\alpha}} + c.c. .$$

Then the Lagrange anchor (4) takes this characteristic to the symmetry

$$X = \int_{\mathbb{R}^{3,1}} X(\Psi)_{\alpha_1 \dots \alpha_{2s}} \frac{\delta}{\delta \varphi_{\alpha_1 \dots \alpha_{2s}}} + c.c., \quad (7)$$

where

$$X(\Psi)_{\alpha_1 \dots \alpha_{2s}} = i^{2s} \partial_{(\alpha_2 \dot{\alpha}_2} \dots \partial_{\alpha_{2s} \dot{\alpha}_{2s}} \overline{\Psi}_{\alpha_1}^{\dot{\alpha}_2 \dots \dot{\alpha}_{2s}}.$$

Applying (7) to the characteristics obtained and classified in [18], we get all the characteristic symmetries. Since the Lagrange anchor is strongly integrable, characteristic symmetries form an infinite dimensional Lie subalgebra in the Lie algebra of all symmetries. This subalgebra was previously unknown. For low spins ($s = 1/2, 1$) the Lie algebra of characteristic symmetries contains a finite dimensional subalgebra which is isomorphic to the Lie algebra of conformal group. The elements of this subalgebra correspond to conserved currents that are expressible in terms of the energy-momentum tensor.

CONCLUSION

In this work, we demonstrate a Poincaré invariant Lagrange anchor for the Bargmann-Wigner equations and apply this anchor for deriving symmetries from conservation laws. The Lagrange anchor, being independent of fields, is strongly integrable. Because of this fact, the symmetries connected with the conservation laws (characteristic symmetries) form an infinite-dimensional subalgebra in the full Lie algebra of symmetries. The algebraic meaning of the subalgebra of characteristic symmetries remains unclear at the moment.

The Lagrange anchor (6) may be used for quantization of the Bargmann-Wigner equations. At the free level the corresponding generalized Schwinger-Dyson equations and probability amplitude is found in [23]. It can also be a good starting point for constructing the Lagrange anchor for Vasiliev's equations and development of a quantum theory of higher-spin interactions.

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